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On the Maximal Modulus of Polynomials on Cantor Sets

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Let $0<p<1/8$ and consider the Cantor set $C^*(p)$ (where $C^*(1/3)$ would be the classical Cantor set). For any sequence $\omega=(\xi_1, \xi_2, ...)$, $\xi_\nu \in C^*(p)$, let $B_n(\omega)=\max_{z \in C^*(p)} \prod_{\nu=1}^n |z-\xi_\nu|$. It is shown that there exists a constant $\theta=\theta(p)$, independent of ω, such that $B_n(\omega)>(\log n)^\theta$ for almost all n (i.e., all except a sequence of density zero). An analogous theorem for the unit circle $C=\{|z|=1\}$ instead of $C^*(p)$ (with "infinitely many" instead of "almost all") was proved before by the author (*Bull. London Math. Soc.* **12**, 1980, 81–88), solving a problem of Erdős.

1. INTRODUCTION

Let $K \subset \mathbb{C}$ be an arbitrary compact subset of the complex plane. Denote by $\mathscr{P}_n(K)$ the set of all polynomials of the form $p_n(z)=\prod_{\nu=1}^n (z-a_\nu)$, with all (not necessarily distinct) zeros a_ν in K ("K-polynomials"). We call $m_n(z) \in \mathscr{P}_n(K)$ a *minimal polynomial of degree n* if $\max_{z \in K} |m_n(z)|$ is minimal with respect to all K-polynomials of degree n. Due to the compactness of K, minimal polynomials of degree n always exist, but are not uniquely determined in general.

The numbers $\max_{z \in K} |m_n(z)|$ are characteristic for the set K and will be denoted by $A_n(K)$ $(n=1, 2, ...)$ in the sequel.

Let $\omega=(\xi_1, \xi_2, ...)$ be an arbitrary sequence of (not necessarily distinct) points in K. With every such sequence ω we associate a sequence of K-polynomials $\{q_n(\omega, z)\}$ by letting $q_n(\omega, z)=\prod_{\nu=1}^n (z-\xi_\nu)$. Let $B_n(\omega, K)=\max_{z \in K} |q_n(\omega, z)|$. We have trivially $B_n(\omega, K) \geqslant A_n(K)$ for all n.

The problem we are going to discuss is, roughly speaking, the following: Does there exist a sequence ω in K such that *all* polynomials $q_n(\omega, z)$

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possess the same approximation quality as the minimal polynomials $m_n(z)$? More exactly (cf. P. Erdös [2]): Does there exist an ω with

$$\limsup_{n\to\infty} \frac{B_n(\omega, K)}{A_n(K)} < \infty? \tag{1}$$

For the unit circle $K = \{|z| = 1\}$ the answer is negative. The author [12] proved that for some numerical constant $\theta > 0$ the relation

$$\frac{B_n(\omega, K)}{A_n(K)} > (\log n)^\theta \tag{2}$$

holds for each ω and infinitely many n. Using the reduction method from [10] he can even show that (2) holds for a subsequence of n's of asymptotic density 1 ("almost all n"). In Oberwolfach, 1980, Loxton announced a considerable improvement of the bound (2): $B_n(\omega, K)/A_n(K) > n^{1/(\log\log n)^\theta}$ holds for some $\theta > 0$ and infinitely many n. This result is close to best possible, since there exists a sequence ω with $B_n(\omega, K)/A_n(K) < n$ for all n.

A general answer to problem (1) for arbitrary K seems to be difficult. However, for Jordan curves satisfying certain smoothness conditions the result (2) may be obtained as well. For domains bounded by Jordan curves (satisfying again certain smoothness conditions) the situation is totally different: the answer to problem (1) is positive.

In this paper we restrict ourselves to considering a certain class of Cantor sets. It turns out to be convenient to split problem (1) into two problems: the separate investigation of the behaviour of the numbers $A_n(K)$ and $B_n(\omega, K)$, respectively, with K satisfying a natural norming condition.

2. Some Potential Theory

2.1. *Potentials.* In this section we list some basic facts from potential theory, necessary both for understanding the problem and obtaining quantitative results. Let $K \subset \mathbb{C}$ be a compact set. Denote by $\mathfrak{M}(K)$ the class of all probability measures on the σ-algebra of Lebesgue measurable subsets of K. The *support* supp μ of a probability measure ("p.m.") $\mu \in \mathfrak{M}(K)$ is the set of all points $z \in \mathbb{C}$ with the property that, for each ε-neighbourhood $N_\varepsilon(z)$, the measure $\mu(N_\varepsilon(z) \cap K)$ is positive. Clearly supp $\mu \subset K$.

Every probability measure $\mu \in \mathfrak{M}(K)$ generates a *logarithmic potential* $U_\mu(z)$, defined by $U_\mu(z) = -\int_K \log|z-\zeta|\, d\mu(\zeta)$.

The potential $U_\mu(z)$ exists for all $z \in \mathbb{C}$ (possibly $U_\mu(z) = \infty$), satisfies the inequality $-\infty < U_\mu(z) \leqslant \infty$ for all $z \in \mathbb{C}$, and is a superharmonic function on $\mathbb{C}$. Outside of K, that means in every subdomain of the complementary set $\mathbb{C}\backslash K$, the potential $U_\mu(z)$ is even harmonic.

We introduce an important topological concept: the *outer boundary* of a compact set K. The complement $\mathbb{C}\backslash K$ is the disjoint union of at most countably many domains, exactly one of which, denoted by G_∞, contains the point ∞. The boundary ∂G_∞ of G_∞ is contained in the boundary ∂K of the set K and called the outer boundary of K.

2.2 *Energy, Capacity, and Equilibrium Distribution.* The *energy* $I(\mu)$ of a p.m. $\mu \in \mathfrak{M}(K)$ is defined by the formula

$$I(\mu) = \int_K U_\mu(z)\, d\mu(z) = -\int_K \int_K \log |z-\zeta|\, d\mu(z)\, d\mu(\zeta).$$

Let $V = \inf_{\mu \in \mathfrak{M}(K)} I(\mu)$. The inequality $-\infty < V \leqslant \infty$ holds. The number e^{-V} is called the *logarithmic capacity* of the set K and denoted by $\operatorname{cap}_l K$. We have $0 \leqslant \operatorname{cap}_l K < \infty$. Logarithmic capacity is known to behave linearly when K is submitted to a homothetic transformation.

A set K has zero capacity if and only if all p.m.'s on K possess infinite energy. From now on we restrict ourselves to sets K being "essential" in the sense of potential theory, namely sets K for which $\operatorname{cap}_l K > 0$ holds. In this case there exists a unique probability measure $\gamma \in \mathfrak{M}(K)$, called the *equilibrium distribution* of K, for which the energy $I(\gamma)$ becomes minimal. The support $\operatorname{supp} \gamma$ of the equilibrium distribution is identical with the outer boundary ∂G_∞ of K.

The equilibrium distribution γ has another characteristic property, even more important for our purposes: the potential $U_\gamma(z)$ is constant "almost everywhere" on K in the following sense. We have $U_\gamma(z) = -\log \operatorname{cap}_l K$ for all $z \in K$ except for a subset of logarithmic capacity zero.

To exclude such exceptional sets we make the additional assumption that K be *regular* in the sense of the Dirichlet problem. Though not defining the concept of regularity, we mention the following facts.

(a) The problem of regularity is considered as solved. There are both necessary and sufficient conditions (N. Wiener) and criteria of practical importance (e.g., Poincaré's cone condition for domains).

(b) The sets K we are dealing with in this paper are known to be regular.

By imposing, if necessary, a suitable homothetic transformation on the set K, we may further assume without restriction that the norming condition $\operatorname{cap}_l K = 1$ holds.

From now on let X be a regular compact set with logarithmic capacity 1. For the equilibrium distribution γ the equality $U_\gamma(z) = 0$ holds for all $z \in K$.

2.3. *Polynomials and Potentials.* With every K-polynomial $p_n(z)=\prod_{\nu=1}^{n}(z-a_\nu)$ we associate a discrete probability measure $\pi_n \in \mathfrak{M}(K)$ by assigning to each zero a_ν with multiplicity k_ν the mass k_ν/n. The distribution π_n generates the potential $U_{\pi_n}(z)=-(1/n)\sum_{\nu=1}^{n}\log|z-a_\nu|$, related to the polynomial $p_n(z)$ by the identity $U_{\pi_n}(z)=-(1/n)\log|p_n(z)|$.

We have

$$\max_{z\in K}|p_n(z)|=\exp(-n\cdot\min_{z\in K}U_{\pi_n}(z)). \tag{3}$$

In this way all problems dealing with the modulus of a polynomial can be translated into the language of potential theory.

The discrete distribution associated with a minimal polynomial $m_n(z)$ is called a *minimal distribution* and will be denoted (although not uniquely determined in general!) by μ_n.

For the characteristic numbers $A_n(K)$ the relation $A_n(K)=\exp(-n\cdot\min_{z\in K}U_{\mu_n}(z))$ is valid in view of (3).

Let us first show that $A_n(K)\geqslant 1$ holds for each $n\in\mathbb{N}$. Let γ be the equilibrium distribution of K. The subsequent identities follow from Fubini's theorem and the fact that the equilibrium potential vanishes identically on K.

$$\int_K U_{\mu_n}(z)\,d\gamma(z)=-\int_K\int_K \log|z-\zeta|\,d\mu_n(\zeta)\,d\gamma(z)=\int_K U_\gamma(\zeta)\,d\mu_n(\zeta)=0.$$

So we have $\min_{z\in K}U_{\mu_n}(z)\leqslant 0$, hence $A_n(K)=\exp(-n\cdot\min_{z\in K}U_{\mu_n}(z))\geqslant 1$. It is known from potential theory (Goluzin [4, Chap. VII]) that the limit $\lim_{n\to\infty}A_n(K)^{1/n}$ exists and is equal to the logarithmic capacity of K, hence equal to 1 in our case.

3. The Limiting Behaviour of Minimal Distributions

A sequence $\{\nu_n\}$ of p.m.'s from $\mathfrak{M}(K)$ is called *weakly convergent* to the p.m. $\nu\in\mathfrak{M}(K)$ ($\nu_n \rightharpoonup \nu$), if $\lim_{n\to\infty}\int_K f\,d\nu_n=\int_K f\,d\nu$ holds for every function f continuous on K.

The following theorem is a generalization of a classical result of Fekete (see, for example, [3]), originally stated for the circle and a certain class of Jordan curves.

Theorem 1. *Let $\{\nu_n\}$ be a sequence of probability measures on K, satisfying the relation $\lim_{n\to\infty}\min_{z\in K}U_{\nu_n}(z)=0$. If the support of the equilibrium distribution γ is all of K (which is equivalent to $K=\partial G_\infty$), then the sequence $\{\nu_n\}$ converges weakly to γ.*

COROLLARY. *From the relation* $\lim_{n\to\infty} A_n(K)^{1/n} = 1$, *mentioned at the end of Section* 2, *we see that Theorem* 1 *is valid for any sequence* $\{\mu_n\}$ *of minimal distributions.*

Proof. (a) Assume that $\{\nu_n\}$ does not converge to γ in the weak sense. Then there exists a subsequence $\{\nu_{n_k}\} \subset \{\nu_n\}$, weakly convergent to a p.m. $\nu \in \mathfrak{M}(K)$ with $\nu \neq \gamma$. The proof of the latter statement runs along a well-known pattern, using separability of the space of functions continuous on K, and the Cantor diagonal process.

(b) Consider the potentials belonging to the distributions ν_{n_k} and ν. We have (Landkof [7, Theorem 3.8]) $U_\nu(z) = \liminf_{k\to\infty} U_{\nu_{n_k}}(z)$ for all $z \in K$ except for a set of capacity 0. Since a set of capacity 0 automatically has equilibrium measure 0, the relation $U_\nu(z) = \liminf_{k\to\infty} U_{\nu_{n_k}}(z)$ holds for γ-almost all $z \in K$. Hence, by the assumption, we have $U_\nu(z) \geqslant 0$ for γ-almost all $z \in K$.

(c) Let $\operatorname{Pos} U_\nu = \{z \in K \mid U_\nu(z) > 0\}$. Because of the uniqueness of the equilibrium distribution we have

$$0 = I(\gamma) < I(\nu) = \int_K U_\nu(z)\, d\nu(z).$$

Hence $\operatorname{Pos} U_\nu$ is nonvoid.

On the other hand, using the relation $\int_K U_\nu(z)\, d\gamma(z) = \int_K U_\gamma(\zeta)\, d\nu(\zeta) = 0$ we deduce that $\gamma(\operatorname{Pos} U_\nu) = 0$.

(d) Let $z_0 \in \operatorname{Pos} U_\nu$. Since $\operatorname{supp} \gamma = K$ by assumption, every ε-neighborhood $N_\varepsilon(z_0) \cap K$ has positive γ-measure. Since on the other hand $\gamma(\operatorname{Pos} U_\nu) = 0$ holds, there exists a sequence of points $z_1, z_2, \ldots$ with $z_k \in K \backslash \operatorname{Pos} U_\nu$ and $\lim_{k\to\infty} z_k = z_0$. From the upper semicontinuity of the logarithmic potential we get $U_\nu(z_0) \leqslant \liminf_{k\to\infty} U_\nu(z_k) \leqslant 0$, contrary to the choice of z_0. This proves the assertion.

We finish this paragraph with two remarks.

1) Without the assumption $K = \partial G_\infty$ the theorem is no longer true. A sequence of minimal distributions need not converge to the equilibrium distribution in this case, even need not converge at all. However, the behaviour of the sequence $\{\mu_n\}$ is not arbitrary. Using the notion of the *balayage* of a p.m. $\mu \in \mathfrak{M}(K)$ onto the outer boundary ∂G_∞ of K ("$\operatorname{bal}_{\partial G_\infty} \mu$," see, for example, Chap. IV in Landkof's book), the following generalization of Theorem 1 is valid (no further condition on K).

THEOREM 1′. *Let* $\{\nu_n\}$ *be a sequence of probability measures on* K, *satisfying the relation* $\lim_{n\to\infty} \min_{z\in K} U_{\nu_n}(z) = 0$. *Then* $\operatorname{bal}_{\partial G_\infty} \nu_n \rightharpoonup \gamma$ holds.

If $K = \partial G_\infty$, the balayage of any p.m. $\nu_n \in \mathfrak{M}(K)$ coincides with ν_n itself, and we obtain Theorem 1 as a special case.

2) Erdős' problem (1) admits of the following potential theoretic interpretation. We assume $K=\partial G_\infty$. There exists a unique "ideal distribution" on K with the property $\min_{z\in K} U_\mu(z)=0$, namely the equilibrium distribution γ itself. For any distribution μ different from γ we have $\min_{z\in K} U_\mu(z)<0$. The greater $\min_{z\in K} U_{\pi_n}(z)$ is for a discrete distribution π_n, the better is the uniform approximation quality of the corresponding polynomial $p_n(z)$ on K. Now Erdös' problem may be regarded as the problem of approximating the equilibrium distribution γ on the one hand by discrete n-point distributions π_n (chosen independently for each n), on the other hand by a sequence of discrete distributions, coming from a single sequence $\omega=(\xi_1, \xi_2, ...)$ of points on K.

4. Cantor-Like Sets

4.1. *On the Behaviour of the Numbers A_n.* Now consider a certain class of linear sets of logarithmic capacity 1 and Hausdorff dimension <1. The estimate of the numbers A_n, almost trivial for the unit circle, causes considerable difficulties. At this point we mention again the well-known relations $A_n \geqslant 1$ and $\lim_{n\to\infty} A_n^{1/n}=1$.

Because of the "restricted mobility" on sets of dimension <1 one might conjecture that the sequence of the A_n's cannot be bounded. The following example shows, however, that at least the relation $\lim_{n\to\infty} A_n=\infty$ need not be true.

Denote by $\tau:\mathbb{C}\to\mathbb{C}$ the complex mapping $\tau(z)=z^2-12$. Apply the inverse τ^{-1} iteratively to the disk $\Gamma_0=\{|z|\leqslant 4\}$ and consider the sequence of sets $\Gamma_k=\tau^{-1}(\Gamma_0)$ $(k=0, 1, ...)$ (see Fig. 1).

The following properties hold.

(i) The sequence $\{\Gamma_k\}$ is decreasing. It is sufficient to show $\Gamma_1\subset\Gamma_0$.

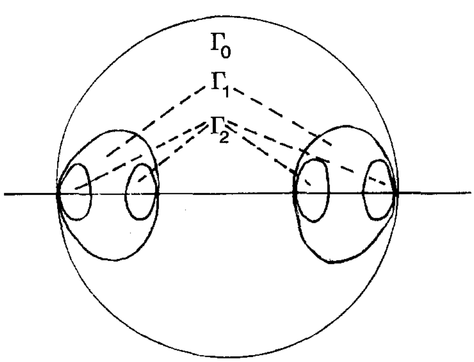


Fig. 1. The shape of the sets Γ_k.

Each point $z_1 \in \Gamma_1$ has a representation $z_1 = \sqrt{z_0 + 12}$ for some $z_0 \in \Gamma_0$. We have $|z_1| \leqslant +\sqrt{|z_0| + 12} \leqslant 4$, hence $z_1 \in \Gamma_0$.

(ii) Each of the sets Γ_k consists of 2^k disjoint connected components. Denote them by $\Gamma_{k\nu}$ $(\nu = 1, 2, ..., 2^k)$.

(iii) The sets Γ_k have logarithmic capacity $4^{1/2^k}$ (see, e.g., Landkof [7, p. 173]).

(iv) The diameter diam $\Gamma_{k\nu}$ of a connected component $\Gamma_{k\nu}$ satisfies the inequality $\operatorname{diam} \Gamma_{k\nu} \leqslant 8 \cdot (2\sqrt{8})^{-k}$. This is because on Γ_0, the inequality $|\tau^{-1}(z)| \leqslant 1/2\sqrt{8}$ holds.

(v) The sets Γ_k are symmetric with respect to the real line.

Let $\Gamma = \bigcap_{k=1}^{\infty} \Gamma_k$. From the right continuity of logarithmic capacity (Landkof [7, p. 139]) we conclude that $\operatorname{cap}_l \Gamma = \lim_{k \to \infty} \operatorname{cap}_l \Gamma_k = \lim_{k \to \infty} 4^{1/2^k} = 1$. By (iv) and (v) Γ is a linear set. In particular we have $\Gamma \subset [-4, 4]$. For the Hausdorff dimension $\dim \Gamma$ the relation $1/3 \leqslant \dim \Gamma \leqslant 2/5$ holds. We define a sequence of Γ-polynomials of degree 2^N $(N = 0, 1, ...)$. Let $p_1(t) = t - \sqrt{8}$ and $p_{2^N}(t) = p_1(\tau^N t)$ $(N = 1, 2, ...)$. We have $\max_{t \in \Gamma} |p_1(t)| = 4 + \sqrt{8}$. Because the mapping τ is onto on Γ, we deduce

$$\max_{t \in \Gamma} |p_{2^N}(t)| = \max_{t \in \Gamma} |p_1(\tau^N t)| = \max_{t \in \Gamma} |p_1(t)| = 4 + \sqrt{8} \qquad \text{for all } N.$$

Hence the relation $A_n(\Gamma) \to \infty$ is not satisfied for the set Γ constructed above. Instead the author conjectures that we may replace the lim by the lim sup for this set and similar ones. In particular, the relation $\lim_{N \to \infty} A_{2^N - 1}(\Gamma) = \infty$ should be true. There is an elementary problem on the unit circle, somewhat related to this latter conjecture, which has been solved by József Beck.

On the unit circle consider polynomials $p_{n-1}(z)$ $(n \geqslant 2)$ of the form $p_{n-1}(z) = \prod_{\nu=1}^{n-1} (z - a_\nu)$, with all the zeros taken from the set of nth unit roots. The author conjecture that $\max_{|z|=1} |p_{n-1}(z)| > n^\theta$ holds for each such polynomial and some numerical constant $\theta > 0$. József Beck, however, disproved it.

THEOREM 2. (J. Beck [1]). *For each degree* $(n-1)$ $(n \geqslant 2)$ *there exists a polynomial* $p_{n-1}(z)$ *of the form described above, with* $\max_{|z|=1} |p_{n-1}(z)| \leqslant c$, *with* $c > 0$ *independent of* n.

There is no immediate transference of Beck's construction to the set Γ, so the problem whether the sequence $\{A_n(\Gamma)\}$ is bounded, remains open.

4.2. *On the Behaviour of the Numbers* $B_n(\omega)$ *for Cantor Sets.* The numbers $B_n(\omega)$ for an arbitrary sequence $\omega = (\xi_1, \xi_2, ...)$ on the set Γ are unbounded. Similar to the case of the unit circle, we can prove that

$B_n(\omega, \Gamma) > (\log n)^\theta$ holds for almost all n and some numerical constant $\theta > 0$. We do not prove the result for this special set Γ, but instead for a whole class of Cantor sets frequently used in mathematics.

Let $0 < p < 1/2$ be a fixed number. Let $q = 1 - 2p$. We construct a monotonically decreasing sequence of sets Q_k $(k = 0, 1, 2, ...)$ in the following way. From the initial set $Q_0 = [0, 1]$ take away an open interval of length $p^0 q$ in such a way that two closed intervals of equal length p remain left. Denote the result by Q_1.

From each of the two separate intervals of the set Q_1 take away an open interval of length $p^1 q$ in such a way that 2^2 closed intervals of equal length p^2 remain left. Denote the result by Q_2.

Continuing the procedure, we obtain a monotonic sequence of sets Q_k, each consisting of 2^k closed intervals of equal length p^k.

The intersection $C = \bigcap_{k=1}^{\infty} Q_k$ is a compact regular set, of Hausdorff dimension $\dim C = \log 2 / \log(1/p)$, and of positive logarithmic capacity for which the inequality $pq \leqslant \operatorname{cap}_l C < 1/4$ holds (Tsuji [11, p. 106]).

For the geometric description of the set C we need a suitable concept. The intersections $C_{k\nu} = C \cap Q_{k\nu}$ $(\nu = 1, 2, .., 2^k)$ are called *k-components* (*or components of order k*) of the set C. Two k-components are called *adjacent* if they are contained in one and the same $(k-1)$-component.

When trying to carry over the ideas of the proofs in [10, 12] to the set C, it turns out that we need independence of the integral $\int_K \log|x-y| \, d\mu(y)$ from the point $x \in C$ for some $\mu \in \mathfrak{M}(C)$. This is the deeper reason why integration has to be with respect to the *equilibrium distribution* γ in order to get numerical results.

For the time being we are working with the set C which is *not normalized yet*. We begin by proving some properties of the equilibrium distribution γ on C.

LEMMA 1. *Let* $0 < p < 1/8$. *Then for any two adjacent k-components* $C_{k\nu_1}, C_{k\nu_2}$ $(k \geqslant 1)$ of the set $C = C(p)$ the following inequality holds:

$$2p \leqslant \frac{\gamma(C_{k\nu_2})}{\gamma(C_{k\nu_1})} \leqslant \frac{1}{2p}. \tag{4}$$

Remark. The assertion of Lemma 1 should be true (with a certain constant $M(p)$ in place of $1/2p$) without the restriction $p < 1/8$, that means for arbitrary $p < 1/2$. Our simple method, however, does not admit a proof of the general case. Loosely speaking, the lemma says that the local variation of the equilibrium distribution γ is not too large.

Proof. To prove (4) we use induction on k.

(i) For $k = 1$ only two 1-components C_{11}, C_{12} exist. For reasons of symmetry we have $\gamma(C_{11}) = \gamma(C_{12})$.

Let (4) be true for $k=1, 2, \ldots, N-1$ and arbitrary adjacent pairs of k-components $C_{k\nu_1}, C_{k\nu_2}$.

(ii) Consider two adjacent N-components, denoted by C_{N1}, C_{N2} without restriction. Assume $\gamma(C_{N1}) \geqslant \gamma(C_{N2})$. Two measures will be defined on the components C_{Ni} $(i=1, 2)$:

(a) The restriction of the equilibrium distribution γ of C onto the components C_{Ni} $(i=1, 2)$. Denote them by γ_{Ni}. Note that the measures γ_{Ni} are no longer normalized.

(b) The equilibrium distributions of the sets C_{Ni} themselves. Denote them by κ_{Ni} $(i=1, 2)$. Since the components C_{Ni} are homothetic to the set C, the probability measures κ_{Ni} may be obtained from γ by a simple linear transformation. Furthermore, we have $\operatorname{cap}_l C_{Ni} = p^N \cdot \operatorname{cap}_l C$.

(iii) Next we choose suitable points $\xi \in C_{N1}$, $\eta \in C_{N2}$. We have

$$\int_{C_{N1}} U_{\gamma_{N1}}(x)\, d\kappa_{N1}(x) = \int_{C_{N1}} U_{\kappa_{N1}}(y)\, d\gamma_{N1}(y) = \gamma(C_{N1}) \cdot \log \frac{1}{p^N \cdot \operatorname{cap}_l C}$$

and similarly

$$\int_{C_{N_2}} U_{\gamma_{N2}}(x)\, d\kappa_{N2}(x) = \gamma(C_{N2}) \cdot \log \frac{1}{p^N \cdot \operatorname{cap}_l C}.$$

Now choose $\xi \in C_{N1}$ in such a way that $U_{\gamma_{N1}}(\xi) \geqslant \gamma(C_{N1}) \cdot \log(1/p^N \operatorname{cap}_l C)$ holds. Correspondingly, choose $\eta \in C_{N2}$ with $U_{\gamma_{N2}}(\eta) \leqslant \gamma(C_{N2}) \cdot \log(1/p^N \operatorname{cap}_l C)$.

(iv) We decompose the set C. The union $C_{N1} \cup C_{N2}$ represents an $(N-1)$-component. Denote by C_{N-1} the $(N-1)$-component adjacent to $C_{N1} \cup C_{N2}$. The union $C_{N1} \cup C_{N2} \cup C_{N-1}$ represents an $(N-2)$-component. Denote by C_{N-2} the $(N-2)$-component adjacent to it, and so on. Then $C = C_{N1} \cup C_{N2} \cup C_{N-1} \cup C_{N-2} \cup \cdots \cup C_1$ is a disjoint decomposition of C.

(v) For the points ξ and η chosen in (ii), the relation $U_\gamma(\xi) - U_\gamma(\eta) = 0$ holds by regularity of C. On the other hand, we have

$$\begin{aligned} U_\gamma(\xi) - U_\gamma(\eta) &= \int_C (\log|\eta - x| - \log|\xi - x|)\, d\gamma(x) \\ &= \left(\int_{C_{N2}} \log|\eta - x|\, d\gamma(x) - \int_{C_{N1}} \log|\xi - x|\, d\gamma(x) \right) \\ &\quad + \left(\int_{C_{N1}} \log|\eta - x|\, d\gamma(x) - \int_{C_{N2}} \log|\xi - x|\, d\gamma(x) \right) \\ &\quad + \sum_{k=1}^{N-1} \int_{C_k} \log \left| \frac{\eta - x}{\xi - x} \right| d\gamma(x) = I_1 + I_2 + I_3. \end{aligned}$$

For the three terms I_1, I_2, and I_3 the following estimates are valid:

(I_1) By the choice of the points ξ and η we have

$$I_1 = \int_{C_{N2}} \log|\eta - x| \, d\gamma(x) - \int_{C_{N1}} \log|\xi - x| \, d\gamma(x)$$

$$= U_{\gamma_{N1}}(\xi) - U_{\gamma_{N2}}(\eta)$$

$$\geqslant (\gamma(C_{N1}) - \gamma(C_{N2})) \cdot \log(1/p^N \operatorname{cap}_l C) \geqslant (\gamma(C_{N1}) - \gamma(C_{N2})) \cdot \log(4/p^N).$$

(I_2) We have $|\xi - x| \leqslant p^{N-1}$ for all $x \in C_{N2}$ and $|\eta - x| \geqslant p^{N-1} q$ for all $x \in C_{N1}$. We get for I_2, $I_2 \geqslant \gamma(C_{N2}) \cdot \log(1/p^{N-1}) - \gamma(C_{N1}) \cdot \log(1/p^{N-1} q)$.

(I_3) From the geometry of the set C we deduce

$$\left| \log \left| \frac{\eta - x}{\xi - x} \right| \right| \leqslant \log\left(1 + \frac{p^{N-k}}{q}\right) \leqslant \frac{p^{N-k}}{q} \qquad \text{for all } x \in C_k.$$

By induction hypothesis we have $\gamma(C_k) \leqslant (1/2p)(1+1/2p)^{N-k-1} (\gamma(C_{N1}) + \gamma(C_{N2}))$. Hence for I_3 the following inequality holds:

$$I_3 \geqslant -(\gamma(C_{N1}) + \gamma(C_{N2})) \cdot \sum_{k=1}^{N-1} \frac{p^{N-k}}{q} \cdot \frac{1}{2p} \cdot \left(1 + \frac{1}{2p}\right)^{N-k-1}$$

$$\geqslant -(\gamma(C_{N1}) + \gamma(C_{N2})) \cdot \frac{1}{2q} \cdot \frac{1}{(1/2) - p} = -\frac{1}{q^2} (\gamma(C_{N1}) + \gamma(C_{N2})).$$

Summing up the inequalities (I_1), (I_2), and (I_3) yields

$$\gamma(C_{N1}) \cdot (\log(4q/p) - (1/q^2)) \leqslant \gamma(C_{N2}) \cdot (\log(4/p) + (1/q^2)).$$

For $p < 1/8$ the factor $(\log(4q/p) - (1/q^2))$ is positive, and the inequality $(\log(4/p) + (1/q^2))/(\log(4q/p) - (1/q^2)) < 1/2p$ holds. This proves Lemma 1.

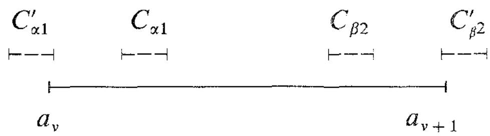
As an immediate consequence of Lemma 1 we obtain a relation between the γ-measure of a component $C_{k\nu} = C \cap Q_{k\nu}$, and the ordinary length of the interval $Q_{k\nu}$. Denote the length of this shortest interval containing the component $C_{k\nu}$ by $|C_{k\nu}|$. We have $|C_{k\nu}| = p^k$.

COROLLARY. *Let* $0 < p < 1/8$. *For any component* $C_{k\nu}$ *the inequality* $|C_{k\nu}| \geqslant (\gamma(C_{k\nu}))^d$ *is valid with* $d = d(p) = \log(1/p)/\log(1 + 2p)$.

Let $0 = a_0 < a_1 < \cdots < a_{n-1} < a_n = 1$ $(n \geqslant 1)$ be an arbitrary decomposition of the unit interval $[0, 1]$. Among the components of C contained in some interval $[a_\nu, a_{\nu+1}]$ $(\nu = 0, 1, ..., n-1)$ we choose certain "largest" components according to the following procedure.

If $(a_\nu, a_{\nu+1}) \cap C \neq \varnothing$ holds, denote by $C_{\alpha 1} \subset [a_\nu, a_{\nu+1}]$ an α-component of minimal order α. The α-component $C'_{\alpha 1}$ adjacent to $C_{\alpha 1}$ cannot be contained in $[a_\nu, a_{\nu+1}]$, since otherwise $C_{\alpha 1} \cup C'_{\alpha 1}$ would be a component of order $(\alpha - 1)$ contained in $[a_\nu, a_{\nu+1}]$, contradicting the minimality of α.

Without restriction we may assume that $C'_{\alpha 1}$ is located left from $C_{\alpha 1}$. Among the components possibly contained in the complementary set $[a_\nu, a_{\nu+1}] \backslash (C_{\alpha 1} \cup C'_{\alpha 1})$, again choose a component $C_{\beta 2}$ of minimal order β. The β-component $C'_{\beta 2}$ adjacent to $C_{\beta 2}$ is located right from $C_{\beta 2}$ and cannot be contained in $[a_\nu, a_{\nu+1}]$.



In this way all the points of $C \cap [a_\nu, a_{\nu+1}]$ are covered, that is, $C \cap [a_\nu, a_{\nu+1}] = (C_{\alpha 1} \cup C'_{\alpha 1} \cup C_{\beta 2} \cup C'_{\beta 2}) \cap [a_\nu, a_{\nu+1}]$. Carrying through the procedure for all intervals $[a_\nu, a_{\nu+1}]$, we obtain in a unique way a set $C_{\alpha 1}, C_{\beta 2}, ...$ of at most $2n$ pairwise disjoint *inner components*. This set of inner components, together with the adjacent components $C'_{\alpha 1}, C'_{\beta 2}, ...$, form a complete (possibly multiple) covering of the set C.

LEMMA 2. *Let* $0 = a_0 < a_1 < \cdots < a_n = 1$ *be a decomposition of the unit interval. Let* $C_{\alpha_1, 1}, C_{\alpha_1, 2}, ..., C_{\alpha_N, N}$ *be the corresponding inner components. Then the inequality* $\sum_{\nu=1}^{N} \gamma(C_{\alpha_\nu, \nu}) \geqslant 2p/(2p+1)$ holds.

Proof. By Lemma 1, for any two adjacent components $C_{\alpha_\nu, \nu}, C'_{\alpha_\nu, \nu}$ the inequality $\gamma(C'_{\alpha_\nu, \nu}) \leqslant (1/2p)\,\gamma(C_{\alpha_\nu, \nu})$ is valid. Since the $C_{\alpha_\nu, \nu}$ and $C'_{\alpha_\nu, \nu}$ together cover the set C, we get

$$1 = \gamma(C) \leqslant \sum \gamma(C_{\alpha_\nu, \nu}) + \sum \gamma(C'_{\alpha_\nu, \nu}) \leqslant \left(1 + \frac{1}{2p}\right) \sum \gamma(C_{\alpha_\nu, \nu}).$$

The result follows.

LEMMA 3. *Let* $C_{k\nu} \subset Q_{k\nu}$ *be an arbitrary component of order* k. *Let* f *be a real-valued function on the interval* $Q_{k\nu}$, *twice continuously differentiable, with its second derivative satisfying the inequality* $-f''(x) \geqslant M > 0$ *in the interior of* $Q_{k\nu}$. *Then the following inequality holds*:

$$\int_{C_{k\nu}} |f(x)|\, d\gamma(x) \geqslant c(p) \cdot M \cdot \gamma(C_{k\nu}) \cdot |C_{k\nu}|^2 \qquad \text{with } c(p) = \frac{8p^6 q^2}{(2p+1)^2}.$$

Proof. (i) The component $C_{k\nu}$ splits into two pairs of adjacent components of order $(k+2)$. We denote them by $C^{(1)}$, $C^{(2)}$ and $C^{(3)}$, $C^{(4)}$, respectively, omitting the index $(k+2)$.

We first assume f to be monotonically non-decreasing on the interval $Q_{k\nu}$. On the component $C_{k\nu}$, we define a "testing function" $u(x)$, piecewise constant, in the following way:

$$u(x)=\begin{cases} -\gamma(C_{k\nu})/\gamma(C^{(1)}) & \text{for } x\in C^{(1)} \\ +\gamma(C_{k\nu})/\gamma(C^{(2)}) & \text{for } x\in C^{(2)} \\ +q\cdot\gamma(C_{k\nu})/\gamma(C^{(3)}) & \text{for } x\in C^{(3)} \\ -q\cdot\gamma(C_{k\nu})/\gamma(C^{(4)}) & \text{for } x\in C^{(4)}\text{.} \end{cases}$$

By Lemma 1, we have $|u(x)|\leqslant(1+1/2p)^2$ for all $x\in C_{k\nu}$. By means of the testing function $u(x)$ the integral $\int_{C_{k\nu}}|f|\,d\gamma$ can be estimated, using the relation

$$\int_{C_{k\nu}}|f|\,d\gamma\geqslant\int_{C_{k\nu}}f(x)\,u(x)\,d\gamma(x)/\sup_{C_{k\nu}}|u(x)|\geqslant\left(1+\frac{1}{2p}\right)^{-2}\int_{C_{k\nu}}f(x)\,u(x)\,d\gamma(x).$$

(ii) We give an estimate for the integral on the right-hand side. We have

$$\int_{C^{(1)}\cup C^{(2)}}u(x)\,d\gamma(x)=\int_{C^{(3)}\cup C^{(4)}}u(x)\,d\gamma(x)=0.$$

Furthermore, by the assumption $f''<0$, f is bounded from above. So we may assume without restriction that $f(x)<0$ holds all $x\in Q_{k\nu}$. Figure 2

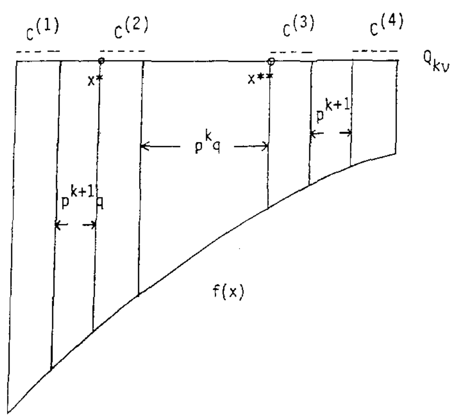


FIGURE 2

illustrates the situation. The marked points x^* and x^{**} are meant to be the left boundary points of the components $C^{(2)}$ and $C^{(3)}$, respectively.

The following estimates result from the mean value theorems:

$$\int_{C^{(1)}\cup C^{(2)}} f(x)\,u(x)\,d\gamma(x) \geqslant \gamma(C_{k\nu})\cdot p^{k+1}q\cdot f'(x^*)$$

and

$$\int_{C^{(3)}\cup C^{(4)}} f(x)\,u(x)\,d\gamma(x) \geqslant -q\cdot\gamma(C_{k\nu})\cdot p^{k+1}\cdot f'(x^{**}).$$

The inequalities together yield

$$\int_{C_{k\nu}} f(x)\,u(x)\,d\gamma(x) \geqslant \gamma(C_{k\nu})\cdot p^{k+1}q\cdot(f'(x^*)-f'(x^{**}))$$

$$\geqslant \gamma(C_{k\nu})\cdot p^{k+1}q\cdot p^k q\cdot M.$$

(iii) If the function f is monotonically non-increasing on $Q_{k\nu}$, a similar argument holds.

Consider the case when f is not monotonic on $Q_{k\nu}$. The component $C_{k\nu}$ splits into two $(k+1)$-components C_{k+1}, C'_{k+1}. By the assumption $f''(x)<0$, the function f is strictly monotonic on at least *one* of these two $(k+1)$-components. Replacing k by $(k+1)$, we may argue in the same way as above.

The three cases together yield the inequality

$$\int_{C_{k\nu}} |f|\,d\gamma \geqslant \left(\frac{2p}{2p+1}\right)^2 \min(p^{2k+1}q^2\cdot\gamma(C_{k\nu})\cdot M,\, p^{2k+3}q^2\cdot\gamma(C_{k+1})\cdot M)$$

$$\geqslant \frac{8p^6q^2}{(2p+1)^2}\cdot M\cdot|C_{k\nu}|^2\cdot\gamma(C_{k\nu}).$$

This proves Lemma 3.

The Cantor sets $C=C(p)$ $(0<p<1/8)$ considered in Lemmas 1–3 possess a logarithmic capacity <1 and have to be normalized first. We do so by applying to C a homothetic mapping with centre in 0 and ray ratio $s=1/\mathrm{cap}_l C$. Thus the set C is transformed into a set C^* with components $C^*_{k\nu}$. The interval $[0, s]$ is the shortest interval containing C^*.

Lemmas 1–3 remain valid with the components $C_{k\nu}$ replaced by $C^*_{k\nu}$.

Let $\omega=(x_1, x_2, \ldots)$ be a sequence of points on C^*. Put $S_n(x,\omega)=\sum_{\lambda=1}^n \log|x-x_\lambda|$. The numbers $B_n(\omega, C^*)$ have been defined previously by

$B_n(\omega, C^*) = \exp(\max_{x \in C^*} S_n(x, \omega))$. The following lemma constitutes the main step used in proving the unboundedness of the sequence $\{B_n(\omega, C^*)\}$.

LEMMA 4. *Let $C^* = C^*(p)$ be a Cantor set with parameter $p < 1/8$ and logarithmic capacity* 1. *Let $l \geqslant 1$, $n \geqslant 1$ be integers, and let* $I = \{i_1, ..., i_u\} \subset \{1, ..., l\}$ and $J = \{j_1, ..., j_v\} \subset \{n + l + 1, ..., n + 2l\}$ *be arbitrary nonvoid index sets. Then there exist constants $c_1 > 0$ and $c_2 > 0$, depending on p only, such that for every sequence ω in C^* the inequality*

$$\int_{C^*} |\max_{j_v \in J} S_{j_v}(x, \omega) - \max_{i_\mu \in I} S_{i_\mu}(x, \omega)| \, d\gamma(x) \geqslant \min\left(c_1, c_2 \cdot \frac{n}{l^{4d}}\right)$$

is valid with $d = d(p) = \log(1/p)/\log(1 + 2p)$.

Proof. For sake of brevity denote by $F(x)$ the function $\max_{j_v \in J} S_{j_v}(x, \omega) - \max_{i_\mu \in I} S_{i_\mu}(x, \omega)$. The function $F(x)$, considered on the interval $[0, s]$, has the following properties:

(i) logarithmic singularities at the points $x_{i_1+1}, ..., x_{j_1}$;

(ii) at most $4l^2$ jump discontinuities of the first derivative;

(iii) the inequality $-F''(x) \geqslant n/s^2$ holds at all points $x \in [0, s]$ where F is twice differentiable;

(iv) let $a, b \in C^*$ $(a < b)$ be two adjacent singularities and let $C^*_{k\nu}$ be an inner component (of order k) belonging to the interval $[a, b]$, then for each $x \in C^*_{k\nu}$ the inequality $-F''(x) \geqslant p^2/|C^*_{k\nu}|^2$ holds. This inequality still holds, if $a = 0$ and only b is a singularity, and if $b = 1$ and only a is a singularity.

The singularities of the function F induce a decomposition of the interval $[0, s]$ into subintervals, which are denoted by h_σ $(\sigma = 1, 2, ...)$. By Lemma 2, the total γ-measure of the inner components belonging to this decomposition has value $\geqslant 2p/(2p + 1)$. Two cases are possible:

(a) The γ-measure of inner components belonging to intervals h_σ *without any jump discontinuity of the first derivative*, is $\geqslant (1/2) \cdot (2p/(2p + 1))$. Applying Lemma 3 and (iv) we get the estimate

$$\int_{C^*} |F(x)| \, d\gamma(x) \geqslant \Sigma' \frac{p^2}{|C^*_{k\nu}|^2} \cdot \frac{8p^6q^2}{(2p+1)^2} \cdot |C^*_{k\nu}|^2 \, \gamma(C^*_{k\nu})$$

$$\geqslant 8(2p + 1)^{-2} \, p^8 q^2 \frac{p}{2p + 1} = c_1(p) > 0.$$

The dash indicates that the sum is taken over inner components $C^*_{k\nu}$, belonging to intervals h_σ without jumps of the first derivative.

(b) The γ-measure of inner components belonging to intervals h_σ *with some jump discontinuity of the first derivative* is $>(1/2)\cdot(2p/(2p+1))$. The number of these h_σ is $\leqslant 4l^2$. Together with the jumps we get $\leqslant 8l^2$ subintervals $H_{\sigma'} \subset \bigcup h\sigma$, on each of which the function $F(x)$ is twice differentiable. Repeating the argument of Lemma 2 we conclude that the total γ-measure of the inner components of the intervals $H_{\sigma'}$ is $\geqslant (p/(2p+1))\cdot(2p/(2p+1)) = 2p^2/(2p+1)^2$.

By Lemma 3 and (iii) we get

$$\int_{C^*} |F(x)|\, d\gamma(x) \geqslant \frac{n}{s^2}\cdot 8(2p+1)^{-2} p^6 q^2 \Sigma' |C^*_{k\nu}|^2 \gamma(C^*_{k\nu}). \tag{5}$$

Here the sum is taken over all inner components of the intervals $H_{\sigma'}$. The number of these inner components is $\leqslant 2\cdot 8l^2$. Applying the Corollary to Lemma 1 to the sum Σ', we get the inequality

$$\begin{aligned}\Sigma' |C^*_{k\nu}|^2\, \gamma(C^*_{k\nu}) &\geqslant \Sigma' s^2 (\gamma(C^*_{k\nu}))^{2d+1} \\ &\geqslant 16 s^2 l^2 \left(\frac{2p^2}{(2p+1)^2\cdot 16 l^2}\right)^{2d+1}.\end{aligned} \tag{6}$$

Substituting (6) into (5) yields

$$\int_{C^*} |F(x)|\, d\gamma(x) \geqslant c_2\cdot\frac{n}{l^{4d}} \qquad \text{with } c_2 = c_2(p) > 0.$$

This proves Lemma 4.

THEOREM 3. *Let* $M \geqslant 0$ *and* $N \geqslant 1$ *be integers. Let* $\mathfrak{A} \subset \{M+1, ..., M+N\}$ *be an index set, and denote by* $|\mathfrak{A}|$ *the cardinality of* $\mathfrak{A}$. *Assume that* $|\mathfrak{A}| \geqslant N\cdot c_3^{\log\log N}$ *for a certain positive constant* $0 < c_3(p) < 1$ *to be specified later. Then for every sequence* ω *on* C^* *the inequality*

$$\int_{C^*} \max_{n_\nu \in \mathfrak{A}} S_{n_\nu}(x, \omega)\, d\gamma(x) \geqslant c\cdot \log\log N$$

holds for some numerical constant $c = c(p) > 0$.

Proof. For each δ with $0 < \delta \leqslant 1$ and every integer $N \geqslant 1$ we define a number $\rho(\delta, N)$ as follows. Put $\rho(\delta, N) = \inf \int_{C^*} \max_{n_\nu \in \mathfrak{A}} S_{n_\nu}(x, \omega)\, d\gamma(x)$, where the inf is taken over *all* sequences ω on C^* and *all* index sets $\mathfrak{A} \subset \{M+1, ..., M+N\}$ with $|\mathfrak{A}| \geqslant \delta\cdot N$.

Since integration is with respect to the equilibrium distribution γ on C^*, we always have $\int_{C^*} S_{n_\nu}(x, \omega)\, d\gamma(x) = 0$, hence $\rho(\delta, N) \geqslant 0$. For the same reason the number $\rho(\delta, N)$ is independent of M.

(I) First let N be of the form $N = n_t = 2^{e^t}$ ($t \geqslant 1$, integer), where $e = e(p)$ is the least integer *greater* than $4d = 4 \cdot (\log(1/p)/\log(1+2p))$. Let further $n_0 = 2$. For an arbitrary index subset $\mathfrak{A} \subset \{M+1, ..., M+n_t\}$ with $|\mathfrak{A}| \geqslant d \cdot n_t$ consider the intersections

$$\mathfrak{A}_\lambda = \mathfrak{A} \cap \{M + \lambda \cdot n_{t-1} + 1, ..., M + (\lambda+1) \cdot n_{t-1}\}$$
$$\left(\lambda = 0, 1, ..., \frac{n_t}{n_{t-1}} - 1; t \geqslant 1\right).$$

Two cases are possible.

Case 1. There exist subsets $\mathfrak{A}_i, \mathfrak{A}_j$ with $j - i \geqslant (1/2) \cdot (n_t/n_{t-1})$, $|\mathfrak{A}_j| \geqslant (1/2)\, \delta \cdot n_{t-1}$, $|\mathfrak{A}_i| \geqslant (1/2)\, \delta \cdot n_{t-1}$. Then

$$\int_{C^*} \max_{n_\nu \in \mathfrak{A}} S_{n_\nu}(x, \omega)\, d\gamma(x)$$
$$\geqslant \int_{C^*} \max_{\mathfrak{A}_i \cup \mathfrak{A}_j} S_{n_\nu}(x, \omega)\, d\gamma(x)$$
$$= \frac{1}{2} \int_{C^*} \max_{\mathfrak{A}_i} S_{n_\nu}(x, \omega)\, d\gamma(x) + \frac{1}{2} \int_{C^*} \max_{\mathfrak{A}_j} S_{n_\nu}(x, \omega)\, d\gamma(x)$$
$$+ \frac{1}{2} \int_{C^*} |\max_{\mathfrak{A}_j} S_{n_\nu}(x, \omega) - \max_{\mathfrak{A}_i} S_{n_\nu}(x, \omega)|\, d\gamma(x)$$
$$\geqslant \rho\left(\frac{1}{2}\delta, n_{t-1}\right) + \frac{1}{2} \int_{C^*} |\max_{\mathfrak{A}_j} S_{n_\nu}(x, \omega) - \max_{\mathfrak{A}_i} S_{n_\nu}(x, \omega)|\, d\gamma(x).$$

Applying Lemma 4 with $n = n_t - 2n_{t-1}$ and $l = n_{t-1}$ we get

$$\int_{C^*} \max_{n_\nu \in \mathfrak{A}} S_{n_\nu}(x, \omega)\, d\gamma(x) \geqslant \rho\left(\frac{1}{2}\delta, n_{t-1}\right) + \frac{1}{2} \min\left(c_1, c_2 \cdot \frac{n_t - 2n_{t-1}}{n_{t-1}^{4d}}\right)$$
$$\geqslant \rho\left(\frac{1}{2}\delta, n_{t-1}\right) + c_4 \qquad (c_4 = c_4(p) > 0). \qquad (7)$$

Case 2. There is no pair $\mathfrak{A}_i, \mathfrak{A}_j$ having the properties required. Then at least one set $\mathfrak{A}_\lambda$ contains $\geqslant (3/2)\, |\mathfrak{A}|\, (n_{t-1}/n_t)$ elements. Hence, in Case 2, we have

$$\int_{C^*} \max_{n_\nu \in \mathfrak{A}} S_{n_\nu}(x, \omega)\, d\gamma(x) \geqslant \rho\left(\frac{3}{2}\delta, n_{t-1}\right). \qquad (8)$$

Taking the infimum over all permissible index sets $\mathfrak{A}$ and all sequences ω on the left-hand side of (7) and (8), we get

$$\rho(\delta, n_t) \geqslant \min\left(\rho\left(\frac{3}{2}\delta, n_{t-1}\right), \rho\left(\frac{1}{2}\delta, n_{t-1}\right) + c_4\right).$$

Note that Case 2 can occur only if the "density" $(3/2)\,\delta$ satisfies $(3/2)\,\delta \leqslant 1$. Repeating the reduction step t times, we obtain

$$\rho(\delta, n_t) \geqslant \min_{\tau}{}^{*}\left(\rho\left(\left(\frac{1}{2}\right)^{\tau}\left(\frac{3}{2}\right)^{t-\tau}\delta, n_0\right) + \tau \cdot c_4\right).$$

The $*$ indicates that in the brackets only terms occur satisfying the inequality

$$\left(\frac{1}{2}\right)^{\tau}\left(\frac{3}{2}\right)^{t-\tau}\delta \leqslant 1.$$

Assuming $\delta \geqslant (3/4)^t$, the latter inequality implies

$$\tau \geqslant t \cdot \frac{\log(9/8)}{\log 3} = c_5 \cdot t.$$

Hence, for $\delta \geqslant (3/4)^t$, the inequality $\rho(\delta, n_t) \geqslant c_5 \cdot t \cdot c_4 \geqslant c_6 \cdot \log\log n_t$ holds. The positive constants c_6, c_4, c_5 depend on p only.

(II) Let $N \geqslant n_1$ be arbitrary. Choose n_t such that $n_t \leqslant N < n_{t+1}$. If there are $\geqslant \delta \cdot N$ elements of $\mathfrak{A}$ in an index segment of length N, there is a subsegment of length n_t containing at least $(1/2)\,\delta \cdot n_t$ elements of $\mathfrak{A}$. Hence $\rho(\delta, N) \geqslant \rho(1/2)\,\delta, n_t) \geqslant c_6 \cdot \log\log n_t \geqslant c \cdot \log\log N$ holds for $\delta \geqslant c_3^{\log\log N}$. This proves Theorem 3.

THEOREM 4. *Let $\omega = (x_1, x_2, ...)$ be a sequence of points on a Cantor set $C^* = C^*(p)$ $(p < 1/8)$. Then the inequality $B_n(\omega, C^*) > (\log n)^{\theta}$ holds for almost all n and some numerical constant $\theta = \theta(p) > 0$.*

Proof. Theorem 4 follows almost immediately from Theorem 3.

Remarks. 1) Again nothing is known about the behaviour of the numbers $A_n(C^*)$, hence nothing about the ratio $B_n(\omega, C^*)/A_n(C^*)$.

2) At an Oberwolfach conference 1980 Loxton [8] announced a considerable improvement of the author's result [12], using a method due to Halász [5]. This method can be used for the Cantor sets $C^*(p)$ as well, by a suitable change of the auxiliary functions occurring in the Riesz products there.

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